Single Scattering Preconditioner Applied to Boundary Integral Equations

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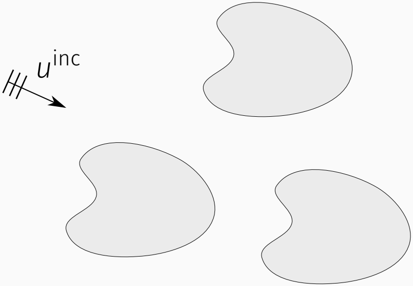
Time-Harmonic Acoustic Multiple Scattering



Boundary Integral Equations

- **Multiple scattering:** coupled problem
- **Large number of obstacles + High frequency** $(\lambda \ll L)$:
  Direct solving hardly possible
- **Indefiniteness:** Slow convergence

Numerical Resolution

Acceleration Techniques

- **Compression** (FMM or H-Matrix): iterative solver
- **Few Algebraic Preconditioners**: SPAI, ...
- **Analytic Preconditioners**: GCFIE, ...

Multiple Scattering Context: Single-Scattering or *Block-Jacobi*

$$\begin{pmatrix} S_{1,1}^{-1} & 0 \\ 0 & S_{2,2}^{-1} \end{pmatrix} \begin{pmatrix} S_{1,1} & S_{1,2} \\ S_{2,1} & S_{2,2} \end{pmatrix} = \underbrace{\begin{pmatrix} Id & S_{1,1}^{-1} S_{1,2} \\ S_{2,2}^{-1} S_{2,1} & Id \end{pmatrix}}_{\mathbb{I}+\mathbb{M}}$$

Related Works

Reflection Method (Block-Jacobi iterative method)

$$X^{n+1} = \mathbb{M}X^n + b$$

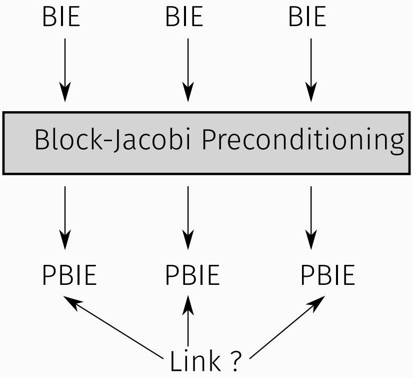
- **Laurent et. al. (2017)** : mathematical framework unification
- **Balabane**: proof and conditions of convergence for Helmholtz (*boundary decomposition method*)

Lax-Foldy Model (approximate the successive reflections)

$$(\mathbb{I} + \mathbb{M})^{-1} \simeq \sum_{\ell=0}^{L} (-\mathbb{M})^{\ell}$$

- **Cassier and Hazard (2013)**: circular obstacles
- **Challa and Sini (2018)**: arbitrary shape obstacles

This talk: *Horizontal* study

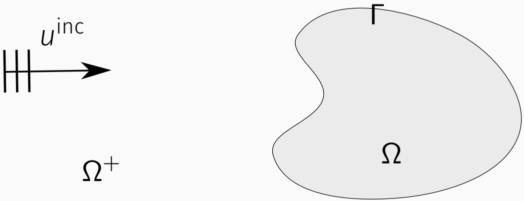


The link

Preconditioned BIEs become *equal* or *similar* (equal up to an invertible operator)

Direct BIE (Null Field Method)

Scattering of an Acoustic Plane Wave



$$\begin{cases} (\Delta + k^2)u = 0 & (\Omega^+) \\ u = 0 & (\Gamma) \\ (u - u^{\text{inc}}) \text{ radiating} & \end{cases}$$

Direct BIE (*Null Field Method*)

- u searched as a **single-layer potential** of density σ

$$u = u^{\text{inc}} + \mathcal{S}\sigma = u^{\text{inc}} + \int_\Gamma G(\cdot, y)\sigma(y)\mathrm{d}y \qquad (\Omega^+)$$

- Fictious wave Inside Ω:

$$\begin{cases} u & = & u^{\text{inc}} + \mathcal{S}\sigma & (\Omega^+) \\ u^- & = & u^{\text{inc}} + \mathcal{S}\sigma & (\Omega) \end{cases}$$

- Jump relations on Γ: $\sigma = \partial_n u^-|_\Gamma - \partial_n u|_\Gamma$
- If $u^- \equiv 0$ in Ω then $\sigma = -\partial_n u|_\Gamma$ (= *Cauchy Data*)

How to make $u^- \equiv 0$?

$$u^- = u^{\text{inc}} + \mathcal{S}\sigma, \quad (\Omega) \iff \begin{cases} (\Delta + k^2)u^- &= 0 & (\Omega) \\ Au^- &= 0 & (\Gamma) \end{cases}$$

A is an **Interior Trace Operator** that **we chose**

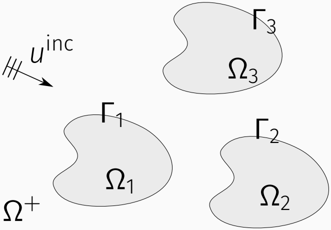
Value of A	Resulting BIE	Name
Trace $\cdot\|_\Gamma$	$S\sigma = -u^{\text{inc}}\|_\Gamma$	(EFIE)
Normal Trace $\partial_n \cdot\|_\Gamma$	$(I/2 - K)\sigma = -\partial_n u^{\text{inc}}\|_\Gamma$	(MFIE)
$\alpha \cdot\|_\Gamma + (1-\alpha)\, \partial_n \cdot\|_\Gamma$	αEFIE $+ (1-\alpha)$MFIE	(CFIE)

with: $S = \mathcal{S}|_\Gamma$ $\quad$ $I/2 - K = \partial_n \mathcal{S}|_\Gamma$

Generic BIE: $(A\mathcal{S}\sigma =)\ S^A\sigma = b^A$ (with $\sigma = -\partial_n u|_\Gamma$)

Multiple Scattering: what change?

BIE in Multiple Scattering Context



Split the potentials

$$u = u^{\text{inc}} + \sum_q \int_{\Gamma_q} G(\cdot, y)\, \sigma|_{\Gamma_q}(y)\mathrm{d}y$$
$$= u^{\text{inc}} + \sum_q \mathcal{S}_q \sigma_q$$

Fictitious field inside Ω:
$$u^- = u^{\text{inc}} + \sum_q \mathcal{S}_q \sigma_q$$

Apply trace operator A:

$$Au^- = 0 \iff \begin{pmatrix} S^A_{1,1} & S^A_{1,2} \\ S^A_{1,2} & S^A_{2,2} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} b^A_1 \\ b^A_2 \end{pmatrix}$$

With: $S^A_{p,q} = A_p \mathcal{S}_q \qquad A_p g = Ag|_{\Gamma_p}$

Single Scattering (or *Block Jacobi*) Preconditioning

$$\underbrace{\begin{pmatrix} \left(S^A_{1,1}\right)^{-1} & 0 \\ 0 & \left(S^A_{2,2}\right)^{-1} \end{pmatrix}}_{\widehat{S}^A} \underbrace{\begin{pmatrix} S^A_{1,1} & S^A_{1,2} \\ S^A_{2,1} & S^A_{2,2} \end{pmatrix}}_{S^A} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} \left(S^A_{1,1}\right)^{-1} b^A_1 \\ \left(S^A_{2,2}\right)^{-1} b^A_2 \end{pmatrix}$$

Single Scattering (or *Block Jacobi*) Preconditioning

$$\underbrace{\begin{pmatrix} I_{1,1} & \left(S^A_{1,1}\right)^{-1} S^A_{1,2} \\ \left(S^A_{2,2}\right)^{-1} S^A_{2,1} & I_{2,2} \end{pmatrix}}_{\widehat{S}^A S^A} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} \left(S^A_{1,1}\right)^{-1} & b^A_1 \\ \left(S^A_{2,2}\right)^{-1} & b^A_2 \end{pmatrix}$$

Proposition [Thierry, 2013]

Consider two trace operators *A and B then* $\widehat{S}^A S^A = \widehat{S}^B S^B$.

i.e. direct BIE becomes exactly the same after block-jacobi preconditioning

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It is enough to show $\left(S^A_{p,p}\right)^{-1} S^A_{p,q} = \left(S^B_{p,p}\right)^{-1} S^B_{p,q}$

Proof of Proposition

Lemma

$$\forall u^{inc}, \qquad \left(S^A_{p,p}\right)^{-1} A_p\, u^{inc} = \left(S^B_{p,p}\right)^{-1} B_p\, u^{inc}$$

Proof: (Single-)Scattering problem in $\Omega_p^+ = \mathbb{R}^3 \setminus \overline{\Omega_p^-}$

$$\begin{cases} (\Delta + k^2)v = 0 & (\Omega_p^+) \\ v = 0 & (\Gamma_p) \\ (v - u^{\text{inc}}) \text{ is outgoing} \end{cases} \iff \begin{cases} v = u^{\text{inc}} + \mathcal{S}_p \mu_p \\ + \begin{cases} S^A_{p,p}\mu_p = A_p g \\ \text{and} \\ S^B_{p,p}\mu_p = B_p g \end{cases} \end{cases}$$

Proof of Proposition

$$\begin{aligned} \forall p \neq q, \quad \left(S^A_{p,p}\right)^{-1} S^A_{p,q} &= \left(S^A_{p,p}\right)^{-1} A_p\, \mathcal{S}_q \\ &= \left(S^B_{p,p}\right)^{-1} B_p\, \mathcal{S}_q \\ &= \left(S^B_{p,p}\right)^{-1} S^B_{p,q} \end{aligned}$$

What about indirect BIE?

Brackage-Werner BIE (BW) vs. Direct BIE (EFIE)

BW and EFIE are **both obtained** by applying the **exterior trace:**

	EFIE	BW
$u = \dots$	$u^{\text{inc}} + \mathcal{S}\sigma$	$u^{\text{inc}} + \mathcal{S}^{BW}\psi$
$u\|_\Gamma = 0$	$S\sigma = -\, u^{\text{inc}}\big\|_\Gamma$	$S^{BW}\psi = -\, u^{\text{inc}}\big\|_\Gamma$

$$\mathcal{S}^{BW} = -\eta\mathcal{S} - \mathcal{D} \qquad S^{BW} = -\eta S - K + I/2$$

Link between the two solutions:

$$S\sigma = S^{BW}\psi \implies \sigma = S^{-1}S^{BW}\psi$$

Precond(BW) vs. Precond(EFIE)

$$\underbrace{\begin{pmatrix} S_{1,1}^{BW} & S_{1,2}^{BW} \\ S_{2,1}^{BW} & S_{2,2}^{BW} \end{pmatrix}}_{S^{BW}} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} b_1^{BW} \\ b_2^{BW} \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} S_{1,1} & S_{1,2} \\ S_{2,1} & S_{2,2} \end{pmatrix}}_{S} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Precond(BW) vs. Precond(EFIE)

$$\underbrace{\begin{pmatrix} \left(S^{BW}_{1,1}\right)^{-1} & 0 \\ 0 & \left(S^{BW}_{2,2}\right)^{-1} \end{pmatrix}}_{\widehat{S}^{BW}} \underbrace{\begin{pmatrix} S^{BW}_{1,1} & S^{BW}_{1,2} \\ S^{BW}_{2,1} & S^{BW}_{2,2} \end{pmatrix}}_{S^{BW}} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \left(S^{BW}_{1,1}\right)^{-1} b^{BW}_1 \\ \left(S^{BW}_{2,2}\right)^{-1} b^{BW}_2 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} \left(S_{1,1}\right)^{-1} & 0 \\ 0 & \left(S_{2,2}\right)^{-1} \end{pmatrix}}_{\widehat{S}} \underbrace{\begin{pmatrix} S_{1,1} & S_{1,2} \\ S_{2,1} & S_{2,2} \end{pmatrix}}_{S} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} \left(S_{1,1}\right)^{-1} b_1 \\ \left(S_{2,2}\right)^{-1} b_2 \end{pmatrix}$$

Precond(BW) vs. Precond(EFIE)

$$\underbrace{\begin{pmatrix} I_{1,1} & \left(S^{BW}_{1,1}\right)^{-1} S^{BW}_{1,2} \\ \left(S^{BW}_{2,2}\right)^{-1} S^{BW}_{2,1} & I_{2,2} \end{pmatrix}}_{\widehat{S}^{BW} S^{BW}} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \left(S^{BW}_{1,1}\right)^{-1} b^{BW}_1 \\ \left(S^{BW}_{2,2}\right)^{-1} b^{BW}_2 \end{pmatrix}$$

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Proposition [Thierry, 2013]

$$\left(S^{-1} S^{BW}\right)^{-1} \widehat{S} S \left(S^{-1} S^{BW}\right) = \widehat{S}^{BW} S^{BW}$$

Precond(BW) $\sim$ Precond(EFIE): Proof

Proposition (recall)

$$\left(S^{-1}S^{BW}\right)^{-1}\widehat{S}S\left(S^{-1}S^{BW}\right)=\widehat{S}^{BW}S^{BW}$$

Reverse engineering:

$$\left(S^{-1}S^{BW}\right)^{-1}\widehat{S}S\left(S^{-1}S^{BW}\right)=\widehat{S}^{BW}S^{BW} \qquad \cdot\times\left(S^{BW}\right)^{-1}$$

$$\left(S^{-1}S^{BW}\right)^{-1}\widehat{S}\,SS^{-1}=\widehat{S}^{BW} \qquad SS^{-1}=I$$

$$\left(S^{-1}S^{BW}\right)^{-1}\widehat{S}=\widehat{S}^{BW} \qquad \text{rearrange}$$

$$\left(S^{BW}\right)^{-1}S\widehat{S}=\widehat{S}^{BW} \qquad S^{BW}\times\cdot$$

$$S\widehat{S}\stackrel{?}{=}S^{BW}\widehat{S}^{BW}$$

Precond(BW) $\sim$ Precond(EFIE): $S\widehat{S} \stackrel{?}{=} S^{BW}\widehat{S}^{BW}$

	$(S\widehat{S})_{p,q}$	$(S\widehat{S}^{BW})_{p,q}$
$p = q$	Id	Id
$p \neq q$	$\left(\mathcal{S}_q S_{q,q}^{-1}\right)\Big\vert_{\Gamma_p}$	$\left(\mathcal{S}_q^{BW} S_{q,q}^{BW,-1}\right)\Big\vert_{\Gamma_p}$

for f_q living on Γ_q, then :

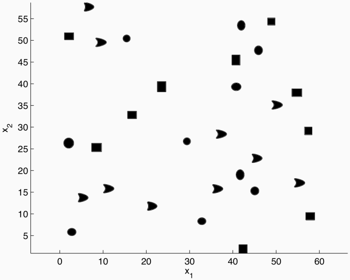
$$\begin{cases} w = \mathcal{S}_q S_{q,q}^{-1} f_q \\ w_{BW} = \mathcal{S}_q^{BW} S_{q,q}^{BW,-1} f_q \end{cases}$$

are radiating solutions of the same single-scattering problem:

$$\begin{cases} (\Delta + k^2)v = 0 \quad (\Omega_q^+) \\ v = f_q \quad (\Gamma_q) \\ v \text{ is radiating} \end{cases}$$

Numerical illustration

Multiple Scattering Problem



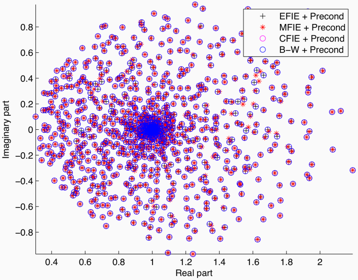
Context

- 2d BEM
- 30 obstacles: size $\simeq 1$
- $\mathrm{dist} \geq 3$
- $k = 20$

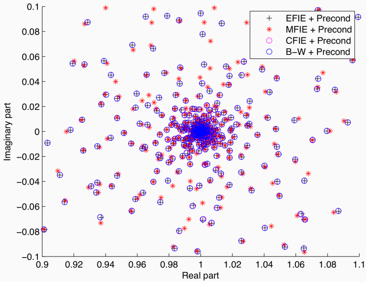
What should we see?

- Same Matrices
- Same Eigenvalues
- Same convergence history of GMRES
- Bonus: *faster convergence*?

Eigenvalues and Matrices



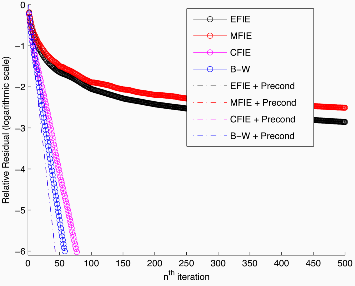
Eigenvalues



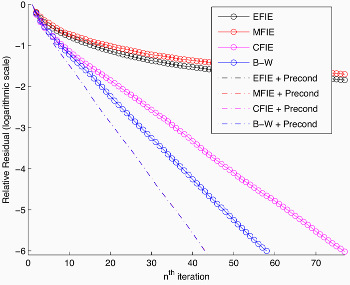
Zoom

- ✓ Relative error between matrices *w.r.t.* $\| \cdot \|_\infty : \simeq 10^{-2}$
- ✓ Maximum relative error between eigenvalues: 2.7%

GMRES: History of Convergence



History of convergence



Zoom

- ✓ Superimposed to each others
- ✓ Faster convergence ...
  - ...But it's a test-case

Conclusion

Conclusion

Block-Jacobi Preconditionner ...

- ✓ *Equalizes* direct BIEs
- ✓ *Similarizes* direct BIEs and BW

  $\longrightarrow$Extended to *every BIE* obtained *via* exterior Dirichlet
- ✓ Numerically illustrated
- ✓ Case of disks + Fourier: Mie Series = direct BIE

What next?

- Maxwell's / Elasticity: (probably) straightforward
- Spectrum of the BIE:

$$\mathbb{I} + \mathbb{M} \text{ with } \mathbb{M} \text{ compact}$$

Thank you!